Solution of HW4

Compulsory Part:

1 Obviously the MC is finite and irreducible, hence it is positive recurrent. Moreover, the chain is aperiodic, hence it has a unique stationary distribution π. Let $\pi = (\pi(0), \pi(1), \pi(2))$, then $\pi P = \pi$ implies that

$$\begin{cases} 0.4\pi(0) + 0.3\pi(1) + 0.2\pi(2) = \pi(0), \\ 0.4\pi(0) + 0.4\pi(1) + 0.4\pi(2) = \pi(1), \\ 0.2\pi(0) + 0.3\pi(1) + 0.4\pi(2) = \pi(2). \end{cases}$$

Together with $\pi(0) + \pi(1) + \pi(2) = 1$, we get $\pi = (\pi(0), \pi(1), \pi(2)) = (0.3, 0.4, 0.3)$.

3 Note that π satisfies $\pi P^m = \pi$ for any positive integer m. Since x leads to y, there is a positive integer n such that $P^n(x, y) > 0$. Hence

$$\pi(y) = \sum_{z \in \mathcal{S}} \pi(z) P^n(z, y) \geq \pi(x) P^n(x, y) > 0.$$

6 The transition matrix is given by

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots \\ q & 0 & p & 0 & 0 & \cdots \\ 0 & q & 0 & p & 0 & \cdots \\ 0 & 0 & q & 0 & p & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Suppose that the stationary distribution π exists. Then by $\pi P = \pi$,

$$\begin{array}{rclcrcl} \pi(1)q & = & \pi(0) & \Rightarrow & \pi(1) & = & \frac{1}{q}\pi(0), \\ \pi(0) + \pi(2)q & = & \pi(1) & \Rightarrow & \pi(2) & = & \frac{\pi(1) - \pi(0)}{q} = \frac{p}{q^2}\pi(0), \\ \pi(1)p + \pi(3)q & = & \pi(2) & \Rightarrow & \pi(2) & = & \frac{\pi(2) - p\pi(1)}{q} = \frac{p^2}{q^3}\pi(0), \\ & & \cdots & & & & \end{array}$$

By induction, $\pi(n) = \dfrac{\pi(0)}{p}\left(\dfrac{p}{q}\right)^n$, $n \geq 1$.

If $p \geq q$ (i.e. $p \geq 1/2$), then $\sum_{n=1}^{\infty} \pi(n) \geq \frac{1}{p} \sum_{n=1}^{\infty} \pi(0) = \infty$. Thus, the stationary distribution does not exist.

On the other hand, if $p < q$ (i.e. $p < 1/2$), we have

$$\sum_{n=0}^{\infty} \pi(n) = \left(1 + \frac{1}{p} \sum_{n=1}^{\infty} \left(\frac{p}{q}\right)^n\right) \pi(0) = \frac{2(1-p)}{1-2p} \pi(0).$$

Hence the unique stationary disrtibution is given by

$$\pi(0) = \frac{1-2p}{2(1-p)}, \quad \pi(n) = \frac{1-2p}{2(1-p)p}\left(\frac{p}{1-p}\right)^n, n \geq 1.$$

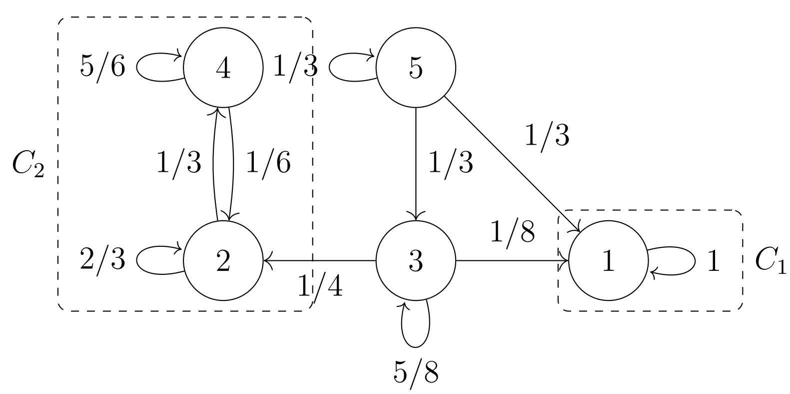
14 Suppose that the stationary distribution π exists. Then $\pi P = P$ and $\sum_{x=0}^{\infty} \pi(x) = 1$ imply that

$$\begin{aligned}
\pi(0) &= \sum_{x=0}^{\infty} \pi(x) P(x,0) = (1-p)\sum_{x=0}^{\infty} \pi(x) = 1-p, \\
\pi(1) &= \pi(0)P(0,1) = (1-p)p, \\
\pi(2) &= \pi(1)P(1,2) = (1-p)p^2, \\
&\cdots
\end{aligned}$$

By induction, $\pi(n) = (1-p)p^n$, $n \geq 0$.

On the other hand, check that above π satisfies both $\sum_{n=0}^{\infty} \pi(n) = 1$ and $\pi(n) = \sum_{m=0}^{\infty} \pi(m)P(m,n)$, $n \geq 0$. Hence $\pi = (1-p, (1-p)p, (1-p)p^2, \cdots)$ is the unique stationary distribution.

AQ The transition diagram of the chain is



for which we can see that $C_1 = \{1\}$ and $C_2 = \{2, 4\}$ are two irreducible closed sets, and $3, 5$ are the transient states.

For convenience, let us permute the matrix into canonical form

$$P_{\text{can}} = \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 3 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 4 & 3 & 5 \end{array} \\ \left[\begin{array}{c|cc|cc} 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 2/3 & 1/3 & 0 & 0 \\ 0 & 1/6 & 5/6 & 0 & 0 \\ \hline 1/8 & 1/4 & 0 & 5/8 & 0 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \end{array}\right] \end{array}$$

We first find the stationary distributions of the chain when restricted to these irreducible closed sets.

- For $C_1 = \{1\}$, since 1 is an absorbing state (with restricted transition matrix $P_1 = \begin{pmatrix}1\end{pmatrix}$), it is easy to see that the stationary distribution is $\pi_1 = (1)$
- For $C_2 = \{2, 4\}$, the transition matrix of the chain when restricted on C_2 is

$$P_2 = \begin{pmatrix} 2/3 & 1/3 \\ 1/6 & 5/6 \end{pmatrix}$$

  We first find the eigenvalues of P_2.
  The characteristic polynomial of P_2 is

$$\det(\lambda I_2 - P_2) = \lambda^2 - \frac{3}{2}t + \frac{1}{2} = (\lambda - \frac{1}{2})(\lambda - 1)$$

  so the eigenvalues are $\lambda = 1$, $\lambda = 1/2$. We can see that 1 is a simple eigenvalue, and all other eigenvalues have modulus less than 1.
  To find the left eigenvector associated to the eigenvalue 1, we solve for the equation

$$\begin{aligned} \pi_2 &= \pi_2 P_2 \\ (I - P_2^\intercal)\pi_2^\intercal &= \pi_2^\intercal \\ \begin{pmatrix} 1/3 & -1/6 \\ -1/3 & 1/6 \end{pmatrix} \pi_2^\intercal &= \pi_2^\intercal \end{aligned}$$

  which we can solve as $\pi_2 = (t, 2t)$ for $t \in \mathbb{C}$. Normalizing the solution, we obtain the stationary distribution $\pi_2 = (1/3, 2/3)$, which has nonnegative entries.

By the theorem from lectures, the limit transition matrices when the chain is restricted to these irreducible closed sets exist, and take the form

$$\lim_{n\to\infty} P_1^n = \begin{pmatrix}1\end{pmatrix}, \quad \lim_{n\to\infty} P_2^n = \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & 2/3 \end{pmatrix}$$

Let us now find the limit transition matrix for the condensed chain, which has the transition matrix

$$\tilde{P} = \begin{array}{c} \\ C_1 \\ C_2 \\ 3 \\ 5 \end{array} \!\!\! \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & 3 & 5 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1/8 & 1/4 & 5/8 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \end{array}\right] \end{array} = \begin{pmatrix} I_2 & 0 \\ R & Q \end{pmatrix}$$

with

$$R = \begin{pmatrix} 1/8 & 1/4 \\ 1/3 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 5/8 & 0 \\ 1/3 & 1/3 \end{pmatrix}$$

For the condensed chain, the fundamental matrix is

$$N = (I - Q)^{-1} = \begin{pmatrix} 8/3 & 0 \\ 4/3 & 3/2 \end{pmatrix}$$

and so

$$NR = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}$$

which implies

$$\lim_{n\to\infty} \tilde{P}^n = \begin{pmatrix} I_2 & 0 \\ NR & 0 \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & 3 & 5 \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1/3 & 2/3 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 \end{array}\right] \end{array}$$

Combined, the limit transition matrix of the original matrix is

$$\lim_{n\to\infty} P^n_{\text{can}} = \begin{array}{c|ccccc} & 1 & 2 & 4 & 3 & 5 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1/3 & 2/3 & 0 & 0 \\ 4 & 0 & 1/3 & 2/3 & 0 & 0 \\ 3 & 1/3 & 2/9 & 4/9 & 0 & 0 \\ 5 & 2/3 & 1/9 & 2/9 & 0 & 0 \end{array}$$

As we permute the original transition matrix into canonical form, we should permute the matrix back into the original order

$$\lim_{n\to\infty} P^n = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 1/3 & 0 & 2/3 & 0 \\ 3 & 1/3 & 2/9 & 0 & 4/9 & 0 \\ 4 & 0 & 1/3 & 0 & 2/3 & 0 \\ 5 & 2/3 & 1/9 & 0 & 2/9 & 0 \end{array}$$

Optional Part:

2 Suppose that the chain has a stationaty distribution π, then it satisfies $\pi P = \pi$, that is, for any $y \in \mathcal{S}$,

$$\pi(y) = \sum_{x\in\mathcal{S}} \pi(x)P(x,y) = \sum_{x\in\mathcal{S}} \pi(x)\alpha_y = \alpha_y.$$

Also one can check that $\pi(y) = \alpha_y, y \in \mathcal{S}$ satisfies

$$\sum_{y\in\mathcal{S}} \pi(y) = \sum_{y\in\mathcal{S}} \alpha_y = \sum_{y\in\mathcal{S}} P(x,y) = 1.$$

Hence $\pi(y) = \alpha_y, y \in \mathcal{S}$ is the unique stationary distribution.

4 Note that π satisfies $\pi P = \pi$. Hence

$$\pi(y) = \sum_{x\in\mathcal{S}} \pi(x)P(x,y) = c\sum_{x\in\mathcal{S}} \pi(x)P(x,z) = c\pi(z).$$

7 **(a)** The transition matrix is given by

$$P = \begin{array}{c} \\ \\ \end{array}\begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & d-2 & d-1 & d \end{array}$$

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{d} & 0 & \frac{d-1}{d} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \frac{2}{d} & 0 & \frac{d-2}{d} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{d} & 0 & \frac{d-3}{d} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & \frac{d-1}{d} & 0 & \frac{1}{d} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

Let π be the stationary distribution. Then by $\pi P = \pi$,

$$\begin{aligned} \pi(1)\frac{1}{d} &= \pi(0) &\Rightarrow \quad \pi(1) &= d\pi(0) = \tbinom{d}{1}\pi(0), \\ \pi(0) + \pi(2)\frac{2}{d} &= \pi(1) &\Rightarrow \quad \pi(2) &= \frac{d(d-1)\pi(0)}{2} = \tbinom{d}{2}\pi(0), \\ \pi(1)\frac{d-1}{d} + \pi(3)\frac{3}{d} &= \pi(2) &\Rightarrow \quad \pi(3) &= \frac{d(d-1)(d-2)\pi(0)}{6} = \tbinom{d}{3}\pi(0), \\ & & \cdots & \end{aligned}$$

By induction, $\pi(n) = \binom{d}{n}\pi(0)$, $0 \le n \le d$. Together with $\sum_{n=0}^{d} \pi(n) = 1$, the stationary distribution must be

$$\pi(n) = \frac{\binom{d}{n}}{2^d}, \quad 0 \le n \le d.$$

(b) The mean of this distribution is given by

$$\sum_{x=0}^{d} x\frac{\binom{d}{x}}{2^d} = \frac{1}{2^d}\sum_{x=0}^{d} x\binom{d}{x} = \frac{d}{2^d}\sum_{x=1}^{d}\binom{d-1}{x-1} = \frac{d}{2^d}2^{d-1} = \frac{d}{2}.$$

Note that

$$\begin{aligned} \sum_{x=0}^{d} x^2\binom{d}{x} &= \sum_{x=2}^{d} x(x-1)\binom{d}{x} + \sum_{x=1}^{d} x\binom{d}{x} \\ &= d(d-1)\sum_{x=2}^{d}\binom{d-2}{x-2} + d\sum_{x=1}^{d}\binom{d-1}{x-1} \\ &= d(d-1)2^{d-2} + d2^{d-1}. \end{aligned}$$

Hence, the variance is given by

$$\sum_{x=0}^{d} x^2\frac{\binom{d}{x}}{2^d} - \left(\sum_{x=0}^{d} x\frac{\binom{d}{x}}{2^d}\right)^2 = \frac{d(d-1)}{4} + \frac{d}{2} - \left(\frac{d}{2}\right)^2 = \frac{d}{4}.$$

8 The transition matrix is given by

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & \cdots & d-2 & d-1 & d \end{array} \\
P = & \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
\frac{1}{2d} & \frac{1}{2} & \frac{d-1}{2d} & 0 & 0 & \cdots & 0 & 0 & 0 \\
0 & \frac{2}{2d} & \frac{1}{2} & \frac{d-2}{2d} & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & \frac{3}{2d} & \frac{1}{2} & \frac{d-3}{2d} & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \cdots & \frac{d-1}{2d} & \frac{1}{2} & \frac{1}{2d} \\
0 & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\end{array}
$$

Let π be the stationary distribution. Then by $\pi P = \pi$,

$$
\begin{aligned}
\pi(1)\frac{1}{2d} &= \frac{\pi(0)}{2} &\Rightarrow \quad \pi(1) &= d\pi(0) = \binom{d}{1}\pi(0), \\
\frac{\pi(0)}{2} + \pi(2)\frac{2}{2d} &= \frac{\pi(1)}{2} &\Rightarrow \quad \pi(2) &= \frac{d(d-1)\pi(0)}{2} = \binom{d}{2}\pi(0), \\
\pi(1)\frac{d-1}{2d} + \pi(3)\frac{3}{2d} &= \frac{\pi(2)}{2} &\Rightarrow \quad \pi(3) &= \frac{d(d-1)(d-2)\pi(0)}{6} = \binom{d}{3}\pi(0), \\
& & \cdots &
\end{aligned}
$$

By induction, $\pi(n) = \binom{d}{n}\pi(0)$, $0 \le n \le d$. Together with $\sum_{n=0}^{d} \pi(n) = 1$, the stationary distribution must be

$$\pi(n) = \frac{\binom{d}{n}}{2^d}, \quad 0 \le n \le d.$$

The result is the same as the one of the original Ehrenfest chain.

9 Let π be the stationary distribution. The transition function is given by

$$
P(x,y) = \begin{cases}
q_x = \left(\frac{x}{d}\right)^2, & \text{if } y = x-1, x \neq 0; \\
r_x = 2\left(\frac{x}{d}\right)\left(\frac{d-x}{d}\right), & \text{if } y = x; \\
p_x = \left(\frac{d-x}{d}\right)^2, & \text{if } y = x+1, x \neq d; \\
0, & \text{otherwise.}
\end{cases}
$$

We can apply the result in page 51 of the textbook, for $x \ge 1$,

$$\pi_x = \frac{p_0 \cdots p_{x-1}}{q_1 \cdots q_x} = \frac{d^2(d-1)^2 \cdots (d-x+1)^2}{(x!)^2} = \binom{d}{x}^2,$$

and set $\pi_0 = 1 = \binom{d}{0}$. By the hint,

$$\pi(0) = \frac{1}{\sum_{x=0}^{d} \pi_x} = \frac{1}{\binom{2d}{d}} = \frac{\binom{d}{0}^2}{\binom{2d}{d}}.$$

Hence $\pi(x) = \pi_x \pi(0) = \frac{\binom{d}{x}^2}{\binom{2d}{d}}$, $0 \le x \le d$.